PMM U.S.S.R.,Vol.45,pp.564-567
0021-8928/82/4 0564 $7.50/0

UDC 532.516

CALCULATION OF CLOSE INTERACTION BETWEEN DROPS, WITH INTERNAL CIRCULATION AND SLIP EFFECT TAKEN INTO ACCOUNT*

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An axisymmetric problem of motion of two spherical drops in a viscous medium is studied in the Stokes approximation. The drop viscosities are assumed finite but large, compared with the viscosity of the surrounding medium. A small degree of slippage is also allowed at the sphere surfaces. An asymptotic solution of the problem is constructed, applicable to the case when the gap between the sphere surfaces is small. In particular cases when slippage or internal circulation are absent, the solution agrees with the results of /1,2/.

The asymptotic solution of the axisymmetric problem constructed in /1/ for the drop viscosities large comapared with the viscosity of the surrounding medium, holds only for very small values of the dimensionless gap (see /1/). The solution constructed here is found to have a wider range of applications. The case when the viscosities of the particles are large compared with the viscosity of the medium, is often encountered when the drops move in a gaseous medium, and in this case the molecular effects must be taken into account when the size of the gap becomes comparable with the free path of the gas molecules. A strict approach based on solving the Boltzman equation does not exist, therefore the present paper uses a model of a flow of continuous medium with slippage at the sphere surfaces which, strictly speaking, holds only for small values of the Knudsen number calculated for the gap size.

1. Formulation of the problem. The liquid spheres have radii a_1 and a_2, dynamic viscosities μ_1 and μ_2, and move along the line drawn through the centers, with velocities $\mathbf{V}_1$ and $\mathbf{V}_2$, through the medium of viscosity μ_e. The Reynolds numbers and the relative velocity of motion of the spheres are assumed small, and the problem is studied within the framework of the quasi-stationary Stokes equations. We use, as the boundary conditions, the absence of flow of the liquids across the sphere surfaces, the continuity of tangential stresses and conditions of slippage

$$\beta_i (\mathbf{v}^e - \mathbf{v}^i)\cdot \tau = \mathbf{p}_n^e \cdot \tau$$

Here τ is a vector tangent to the surface of the sphere of radius a_i, β_i are the slippage coefficients, $\mathbf{p}_n$ and $\mathbf{v}$ are the liquid stress and velocity vectors and n is the direction of the outward normal to the sphere surface. The superscripts e, i $(i = 1, 2)$ denote the quantities in the region between the spheres and inside the sphere of radius a_i respectively. The surface tension at the interface between the liquids is assumed to be sufficiently large, and this makes it possible to neglect the deviation of the particles form from the spherical and omit from our discussion the boundary condition of the continuity of normal stresses.

Since the problem is linear, we can write the hydrodynamic forces in the form

$$\mathbf{F}_1 = -6\pi\mu_e a_1 [\Lambda_{11} (\mathbf{V}_1 - \mathbf{V}_2) + \Lambda_{12}\mathbf{V}_2]$$
$$\mathbf{F}_2 = -6\pi\mu_e a_2 [\Lambda_{21} (\mathbf{V}_2 - \mathbf{V}_1) + \Lambda_{22}\mathbf{V}_2], \quad \Lambda_{11} = k^{-1}\Lambda_{21} = \Lambda_{12}, \quad k = a_1 / a_2 \tag{1.1}$$

The last condition of (1.1) follows from the reciprocity theorem /3/ and the boundary conditions. The viscosity of the drops is assumed constant but large, compared with the viscosity of the medium, and the slippage is assumed small

$$\lambda_i = \mu_i / \mu_e \gg 1, \quad \mu_e / (\beta_i a_i) \ll 1, \quad i = 1, 2 \tag{1.2}$$

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*Prikl.Matem.Mekhan.,45,No.4,759-763,1981

In the case of drops moving through a gaseous medium, the slippage coefficients are given by $\mu_e / \beta_i = A_i l$. Here l is the length of free path of the gas molecules. The coefficients A_i are of the order of unity, and connected in the usual manner with the accommodation coefficients. Thus the last condition of (1.2) implies the smallness of the Knudsen number calculated for the sphere radii. Since the coefficients Λ_{12}, Λ_{22} remain finite when the spheres are in contact, it follows that the values of these coefficients in (1.2) differ little, right up to the moment when the spheres touch, from the corresponding values Λ_{12}^s, Λ_{22}^s for solid spheres calculated in /4/ for the case when the slippage effects are absent. Therefore it is sufficient to put $\mathbf{V}_2 = 0$ and investigate the asymptotics of Λ_{11} for small gaps, with (1.2) taken into account.

2. General structure of the exact solution. First we shall establish the general structure of the exact solution obtained without assuming that the gap is small and (1.2) hold. We pass from the cylindrical coordinates z, ρ (the z-axis is directed along the lines connecting the sphere centers, from sphere a_2 towards sphere a_1, and the ρ-axis is perpendicular to the line connecting the centers, to the bispherical coordinates

$$z = \frac{c \operatorname{sh} \eta}{\operatorname{ch} \eta - \mu}, \quad \rho = \frac{c \sin \xi}{\operatorname{ch} \eta - \mu}, \quad \mu = \cos \xi, \quad 0 \leqslant \xi \leqslant \pi$$

The parameter c and the quantities $\eta_1 > 0$, $\eta_2 < 0$ are determined so that the sphere of radius a_i is a surface $\eta = \eta_i = \text{const}$, and we do it by setting

$$\operatorname{ch} \eta_1 = \frac{(1+\varepsilon)(1+k) + k\varepsilon^2/2}{1+k+k\varepsilon}, \quad \operatorname{sh} \eta_2 = -k \operatorname{sh} \eta_1, \quad c = a_1 \operatorname{sh} \eta_1 \tag{2.1}$$

where εa_1 denotes the gap between the sphere surfaces. Using the general solution /5/ of the Stokes equation is bispherical coordinates, we shall seek the stream function in the form

$$\Psi = -2^{-1/2} V_1 c^2 (\operatorname{ch} \eta - \mu)^{-3/2} \sum_{n=1}^{\infty} n(n+1) \varphi_n(\eta) Q_n(\mu) \tag{2.2}$$

$$\varphi_n^e(\eta) = E_n \operatorname{ch}(n - 1/2)\eta + F_n \operatorname{sh}(n - 1/2)\eta + G_n \operatorname{ch}(n + 3/2)\eta + H_n \operatorname{sh}(n + 3/2)\eta, \quad \eta_2 \leqslant \eta \leqslant \eta_1$$

$$\varphi_n^i(\eta) = A_n^i \exp[-(n - 1/2)|\eta|] + B_n^i \exp[-(n + 3/2)|\eta|], \quad |\eta| \geqslant |\eta_i|, \quad i = 1, 2$$

Here V_1 is the projection of the velocity $\mathbf{V}_1$ on the z-axis, and $Q_n(\mu)$ are the Gegenbauer polynomials. Let us put $\varphi_n^*(\eta) = \varphi_n(\eta)$ near the sphere $\eta = \eta_2$ and $\varphi_n^*(\eta) = \varphi_n(\eta) - R_n(\eta)$ near the sphere $\eta = \eta_1$, where

$$R_n(\eta) = \frac{\exp[-(n + 3/2)\eta]}{2n+3} - \frac{\exp[-(n - 1/2)\eta]}{2n-1} \tag{2.3}$$

Then the boundary conditions will assume the form (the plus sign corresponds to $i = 1$)

$$\varphi_n^{*e} = \varphi_n^{*i} = 0, \quad d^2\varphi_n^{*e} / d\eta^2 = \lambda_i \, d^2\varphi_n^{*i} / d\eta^2 \tag{2.4}$$

$$\frac{\mu_e(\operatorname{ch}\eta - \mu)}{c\beta_i} \sum_{n=1}^{\infty} \frac{d^2\varphi_n^{*e}}{d\eta^2} n(n+1) Q_n(\mu) + \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_i(2n+1)} \frac{d^2\varphi_n^{*e}}{d\eta^2} \pm \frac{d\varphi_n^{*e}}{d\eta} \right) n(n+1) Q_n(\mu) = 0, \quad \eta = \eta_i, \quad i = 1, 2$$

Using the second relation of (2.4), we reduce the boundary condition of slippage to an equation containing no internal functions $\varphi_n^i(\eta)$, and this appears to be important in determining the asymptotics of the solution. The conditions (2.4) can be reduced to two difference equations in e.g. E_n and F_n. Solving the equations obtained in this manner we find, that the stream function can be determined in all regions of flow. In the case of solid spheres with boundary slippage conditions, an exact solution was obtained in /6/.

3. Asymptotic solution. Let us find the inner expansion for the stream function Ψ^e valid in the region of the small gap separating the sphere surfaces, since the region determines the singular part of Λ_{11}. The case of contact between the spheres corresponds to the passage to the limit

$$\varepsilon, \ \eta_1, \ \eta_2 \to 0 \tag{3.1}$$

Let us set

$$\alpha_i = \mu_e/(\varepsilon a_1 \beta_i), \quad p_i = \lambda_i \sqrt{2\varepsilon(1-k)}, \quad i = 1, 2 \tag{3.2}$$

and assume that the quantities k, α_i and p_i are fixed in the limiting passage (3.1). Introducing the variable $\sigma = (\eta/\eta_1 + k)/(1 + k)$, we note that the inner region corresponds to $\sigma \sim 1$, $1 - \mu \sim 1$. Fixing the value of σ, replacing the expression for $\psi_n^e(\eta)$ in (2.2) by the equivalent differential equation and applying the passage to the limit (3.1) to the expressions (2.3) and (2.4), we find

$$\Psi^e \simeq -V_1 a_1^2 \eta_1^2 2^{-1/2} (1-\mu)^{-3/2} \sum_{n=1}^{\infty} n(n+1) \Psi_n(\sigma) Q_n(\mu) \tag{3.3}$$

$$\Psi_n(\sigma) = A_n \sigma^3 + B_n \sigma^2 + C_n \sigma + D_n$$

$$\Psi_n(1) = -4(2n-1)^{-1}(2n+3)^{-1}, \quad \Psi_n(0) = 0$$

$$\alpha_i (1-\mu) \sum_{n=1}^{\infty} n(n+1) \Psi_n''(\sigma_i) Q_n(\mu) + 2 \sum_{n=1}^{\infty} n(n+1) \left[\frac{\psi_n''(\sigma_i)}{p_i(2n+1)} \pm \psi_n'(\sigma_i) \right] Q_n(\mu) = 0, \quad \sigma = \sigma_i, \quad i = 1, 2$$

$$(\sigma_1 = 1, \; \sigma_2 = 0)$$

The above equations yield two difference equations for A_n and B_n

$$\alpha_1 \left[3A_n + B_n - \frac{n-1}{2n-1}(3A_{n-1} + B_{n-1}) - \frac{n+2}{2n+3}(3A_{n+1} + B_{n+1}) \right] + \frac{6A_n + 2B_n}{p_1(2n+1)} + 2A_n + B_n = \frac{4}{(2n-1)(2n+3)} \tag{3.4}$$

$$\alpha_2 \left[B_n - \frac{n-1}{2n-1} B_{n-1} - \frac{n+2}{2n+3} B_{n+1} \right] + \frac{2B_n}{p_2(2n+1)} + A_n + B_n = -\frac{4}{(2n-1)(2n+3)}$$

The behavior of the first term of the inner expansion for Ψ^e in the region where it merges with the external expansion, i.e. when $\mu \to +1$, is determined by the behavior of the coefficients A_n, B_n and C_n as $n \to \infty$. Since the external expansion in the first approximation is the same as that for the solid spheres without slippage, we must demand that

$$A_n \sim 2n^{-2}, \quad B_n \sim -3n^{-2} \tag{3.5}$$

as $n \to \infty$. The conditions (3.5) do not contradict the system (3.4), and yield a unique solution.

According to /1/, the contribution of the inner region towards the magnitude of the force F_1^z is, in the first approximation, equal to

$$-\frac{\pi\mu_e}{c} \int_{-1}^{1} (1-\mu) \frac{\partial^3 \Psi^e}{\partial \eta^3} \, d\mu$$

Taking into account (2.1) and (3.3), we finally obtain

$$\lambda_{11} \simeq (1+k)^{-2} \varepsilon^{-1} f(\alpha_1, \alpha_2, p_1, p_2) \qquad f = 4 \sum_{n=1}^{\infty} \frac{n(n+1) A_n}{(2n-1)(2n+1)(2n+3)} \tag{3.6}$$

In the particular case when slippage is absent $(\alpha_1 = \alpha_2 = 0)$, we can obtain from (3.4) an explicit expression for A_n. As a result we obtain

$$f = 32 \sum_{n=1}^{\infty} \frac{n(n+1)[(2n+1) p_1 p_2 + p_1 + p_2]}{(2n-1)^2 (2n+3)^2 [(2n+1)^2 p_1 p_2 + 4(2n+1)(p_1 + p_2) + 12]} \tag{3.7}$$

Expanding the general term of the series (3.7) into partial fractions, we can express f in terms of the logarithmic derivative $\psi(z)$ of gamma function. When $p_1 = p_2 = p$, we have (γ is the Euler constant)

$$f = \frac{(39 + 7p) p^2}{6(p+3)^2 (p-3)} + \frac{9\pi^2 p}{16(9 - p^2)} + \frac{p^2 (p^2 - 36)}{4(p+3)^2 (p-3)^2} \left[\psi\left(\frac{3}{2} + \frac{3}{p}\right) - \frac{8}{3} + \gamma + 2 \ln 2 \right] \tag{3.8}$$

When $p_1 \neq p_2$, the corresponding expression is very bulky, and is therefore omitted. Using the asymptotics of $\psi(z)$ as $z \to \infty$ we find, for small p_1, p_2 $(p_1 \sim p_2)$,

$$f \simeq \frac{1}{32} \pi^2 (p_1 + p_2) + \frac{1}{9} (p_1^2 + p_2^2 - p_1 p_2) \ln p_1 \tag{3.9}$$

Taking into account (1.2) and (3.2) we find, that the formulas (3.6) and (3.9) agree with the results obtained in /1/ for the moderate values of λ_1 and λ_2. As $p_1, p_2 \to \infty$, the formula (3.7) yields $f = 1$, which agrees with the asymptotic solution /7/ for the solid spheres. The values of Λ_{11}^* obtained from the formulas (3.6) and (3.8) are compared below with the exact values of Λ_{11} for $k = 0.5$, $\lambda_1 = \lambda_2 = 10$ and various ε

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Λ^*_{11}	3.33	21.8	106	412
Λ_{11}	4.80	23.8	109	415

In the absence of internal circulation $(p_1 = p_2 = \infty)$ the relations (3.3) yield

$$\frac{\partial^3 \Psi^e}{\partial \varsigma^3} \simeq \frac{3V_1 c^2 (1+\mu)[2 + (\alpha_1+\alpha_2)(1-\mu)]}{(1-\mu)[3\alpha_1\alpha_2(1-\mu)^2 + 2(\alpha_1+\alpha_2)(1-\mu)+1]}$$

Integrating we obtain

$$f = -\frac{(\alpha_1+\alpha_2)}{6\alpha_1\alpha_2} + \frac{\varphi(t_1)-\varphi(t_2)}{12\alpha_1\alpha_2(t_2-t_1)}, \quad t_{1,2} = \frac{\alpha_1+\alpha_2 \mp \sqrt{\alpha_1^2+\alpha_2^2-\alpha_1\alpha_2}}{3\alpha_1\alpha_2} \tag{3.10}$$

$$\varphi(t) = (2+t)[2-(\alpha_1+\alpha_2)t]\ln[t^{-1}(2+t)]$$

The relations (3.6) and (3.10) generalize the results of /2/ to the case of different α_1, α_2.

Table 1

α	$p=\infty$	8	6	4	3	2	1
0	1000	813	766	687	623	526	361
0.1	844	711	675	614	564	484	342
0.2	743	639	610	561	519	451	325
0.3	668	584	560	518	482	424	311
0.4	611	540	520	484	453	401	299
0.5	566	504	487	455	427	381	288

When $\alpha_1, \alpha_2 \to 0$, (3.10) yields $f = 1$ which corresponds to the first term of the asymptotics /7/. In the general case the system (3.4) has to be solved by numerical methods. The Table 1 gives the values of $f \times 10^3$ for $\alpha_1 = \alpha_2 = \alpha$ and $p_1 = p_2 = p$.

The author thanks A.M. Golovin for the interest shown.

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Translated by L.K.